Abstract

The per-chip capacity of a CDMA communication channel with a varying number of users is calculated. It was assumed that users transmit using BPSK modulation and that the receiver achieves chip-synchronous decoding; therefore inter-symbol interference was ignored. In such a channel, CDMA is characterized by its chip-wise operation, which can be accurately modeled as an additive channel. The per-chip capacity was first determined assuming equal user powers and was shown to increase logarithmically with the number of users. Capacity was also determined for unequal user powers, which showed an increase in capacity relative to the equal user power case. Further, this capacity gain is seen to decrease exponentially with decreasing SNR. Finally a numerical analysis is presented, showing that unbiased input symbols maximize capacity.

Acknowledgments

Special thanks to professor Bajscy and Mr. Beainy for leading me through this project.

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INTRODUCTION

To say that wireless multi-user communications systems will soon come to pervade our every day lives has become a truism. Indeed, the number of cellular phones is multiplying at an almost exponential rate, hand-held devices that are able to access the Internet are emerging, and wireless-based navigation systems for automobiles will soon become standard. The resources available in terms of channel capacity are less and less plentiful relative to demand. The specific access technology that has the limelight is CDMA (Code Division Multiple Access), achieving a far greater spectral efficiency than systems utilizing traditional access technology such as TDMA and FDMA. In addition, CDMA is an easy-going technology in the respect of allowing multiple concurrent transmissions on the same bandwidth. However in the face of the consequent competition for access, capacity must be allocated with a lot of care. The capacity of CDMA in terms of number of possible users in a given bandwidth or equivalently the number of information bits that can be sent per time-slot will be calculated in both a noise-free and a noisy environment.

1 BASIC CONCEPTS OF CDMA

This section will introduce some of the basic concepts relating to the functioning and implementation of Code Division Multiple Access (CDMA) Systems. Readers that are already familiar with this topic are referred to the next section.

1.1 BANDWIDTH MANAGEMENT

The two multiple access techniques that have been used and are still somewhat *en vogue* are FDMA and TDMA. Both of these techniques use the principle of orthogonality. FDMA uses orthogonality in the frequency domain, assigning a frequency slot to every user and thus subdivided the available bandwidth among the users. TDMA uses orthogonality in the time domain whereby each user is assigned a different time slot and thus has the availability of the entire bandwidth for communication purposes in the allocated time. Both of these techniques are proving to be more and more limiting in terms of capacity and performance.

Indeed, TDMA has the incredible constraint that most of the time users are simply waiting for their turn in order to transmit. Therefore every user wastes a great amount of time when they could be transmitting instead.

Similarly, FDMA is essentially limited by the simple fact the available bandwidth is subdivided among the users into separate channel. Once a channel is occupied, no other user can access this channel until the communication is terminated. If in a given network each channel is used once only then the total system capacity will be limited to the number of channels available, an overwhelming constraint. Therefore FDMA networks are organized in cells[1] where in each cell a set of frequency is used. For cells that are sufficiently distanced, so that signals from one cell reach the other with sufficient attenuation the same set of frequency may be re-used. Traditionally networks utilize a 7-way frequency reuse pattern shown below where each color represents a set of frequency.

[1] A cell is generally seen as geographical area

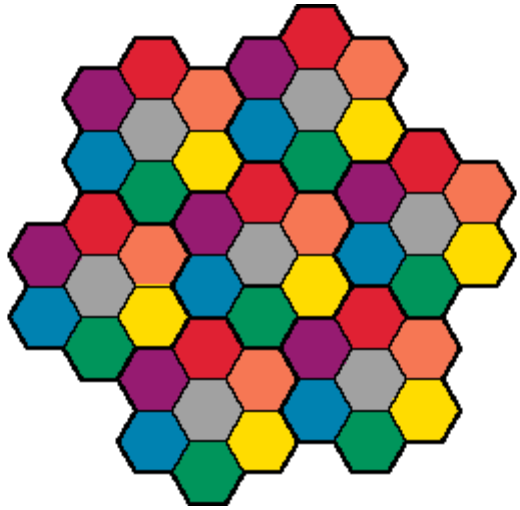

Figure 1. Frequency reuse pattern for traditional cellular systems

That is, cells utilizing the same set of frequencies must be placed sufficiently far apart to account for the worst possible case of interference. Thus the worst-case interference scenario puts is the main factor restricting channel re-use.

CDMA has a radically new approach towards multiple accesses. Instead of partitioning the time or the frequency domain, it allows all the users to talk at the same time and on the entire bandwidth of the channel. For this to be accomplished each user's signal is coded in such a way so that it's spectrum is spread on the bandwidth of the channel. The users' signals are then all transmitted at the same time. This technique works to the extent that the signal are coded and spread so that each user looks like noise to every-body else. It becomes then up to the receiver to extract from the jumble of information that it receives the desired signal. Since all the users are transmitting on the same wide-band channel, the resulting frequency reuse becomes universal and is shown below in figure 2.

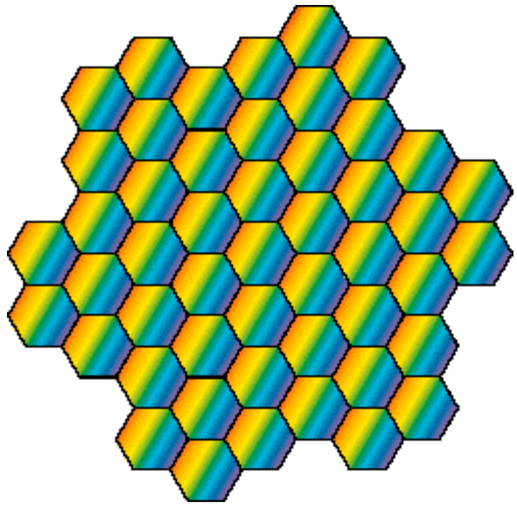

Figure 2. Frequency reuse for CDMA

We should note that the interference experienced by user j is given by the sum of all the other users' powers divided by the channel bandwidth say *W* and can be expressed as follows:

$$I_j = \frac{1}{W}\sum_{i \neq j} P_i$$

Therefore it can be seen that CDMA capacity and performance is sensitive to the average interference. It has been found that system capacity with CDMA is increased between 10 and 20 times when compared with FDMA or TDMA.

1.2 Practical Implementation of CDMA

1.2.1 SPREAD SEQUENCES

We will assume here that BPSK modulation is used. This simply amounts to the restriction that a user is limited to sending +1 and –1. The essence of the CDMA multiple access technique lies in representing a +1 by a vector of such values and –1 by the opposite vector. For example for a particular user we will get the following representation:

$$1 \longrightarrow [1 \;\; -1 \;\; 1 \;\; -1]$$

$$-1 \longrightarrow [-1 \;\; 1 \;\; -1 \;\; 1]$$

The first of these vectors is referred to a spread sequence for the particular user and the components of these vectors or the coordinate are referred to as chips[2]. The effect of implanting a spread sequence is equivalent to a code with rate R=1/4. Therefore to maintain the same information rate, we must quadruple the transmission rate. Since the transmission rate is directly proportional to the bandwidth, the latter is also quadrupled and hence the signal is spread.

Each user in CDMA will be assigned a unique spread sequence distinguishing him from all the other. Instead of sharing time slots or frequency slots, CDMA users share code hence the name of this multiple access technique: Code Division Multiple Access.

1.2.2 ORTHOGONALITY

We recall the assumption that we are using BPSK modulation, and add further to that the assumption that all users are transmitting with the same power and that there is no noise in our system. The important question of when will the receiver be able to extract each user's signal from the received signal arises. Indeed, since all users are transmitting at the same time, the received signal will be the sum performed chip wise or time slot wise. This is shown in the figure 3, with 2 users for simplicity. Note C_i represents the i^{th} chip of the spread sequences

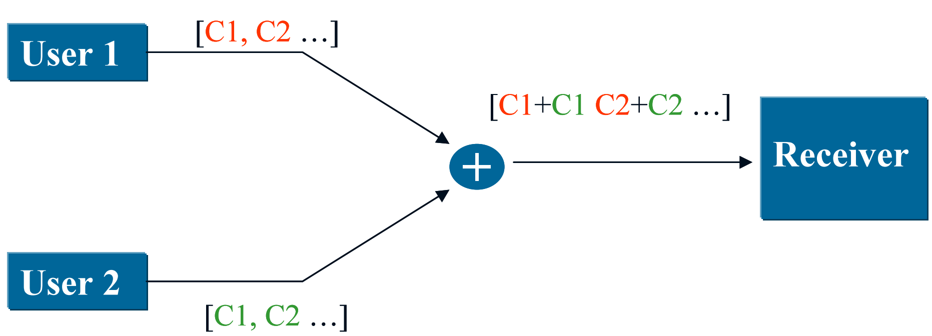

Figure 3. CDMA system with 2 users

We consider in the following analysis that the spread sequences are real-valued vectors: each chip in a spread sequence will represent a coordinate in the corresponding vector.

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[2] A chip is sent in a time slot so these two terms can and will be used interchangeably.

Let R represent the received vector, let b_i represent the sent bit of user i and let S_i represents the spread sequence vector, then R is given by:

$$R = \sum_i b_i S_i$$

Thus the received vector is simply a linear combination of the spread sequences. Now if the spread sequences are orthogonal, they then form a basis, and R is uniquely determined by a combination of the b_i's or equivalently, we may recover the b_i's by projecting R onto the S_i's:

$$b_i = R \cdot S_i$$

Therefore considering the assumptions made in the beginning of this section, we may conclude that with orthogonal spread sequences it is possible to recover each user's sent bit by looking at the entire received vector and projecting it onto the spread sequence vectors. A matched filter at the receiver easily accomplishes this operation.

The requirement that the spread sequences be orthogonal is not at all necessary in a noise free environment and may be relaxed to the condition that the spread sequences be linearly independent. In this case, the S_i's still form a basis and the preceding argument will hold. However in a noisy environment, the orthogonality of the spread sequences provides the most distance – in the Euclidean sense – between the possible received vectors and will thus provide more noise robustness.

1.2.3 IMPRACTICALITY OF ORTHOGONALITY

Although orthogonality will allow for optimal detection at the receiver, it is extremely impractical for several reasons. The first and most obvious reason is that for a system with K users, orthogonality among all users can only be achieved using vectors of dimensionality K or correspondingly spread sequences of length K which is often a large number. Another disadvantage is that users must transmit synchronously to maintain orthogonality.

Furthermore by using orthogonal sequences it is necessary to look K chips – the entire length of the received sequence – in order to extract the K respective bits of each user. Hence the capacity of the CDMA with orthogonal spread sequences is 1bit/chip or 1 bit/time slot. This is the same capacity as that of a TDMA system and our results, which we present later, will show that this is in fact a sub-optimal use of capacity.

2 INFORMATION THEORY

Information Theory gives us the important tools useful to compute the maximum capacity of a given communications system. By this, we mean the maximum amount of information bits that can be transmitted per channel use. Before introducing Shannon's Channel Coding Theorem, which basically provides us with the means of computing the capacity of a channel, we will introduce some basic concepts of Probability Theory.

2.1 Entropy, Conditional Entropy and Mutual Information

Let X, Y denote two arbitrary continuous random variables on (-∞,∞). The *entropy* and *conditional entropy* are given by the equations below respectively:

$$H(Y) = -\int_{-\infty}^{\infty} f_Y(y) \log\{f_Y(y)\} dy \quad \textbf{Equation 1}$$

$$H(Y \mid X) = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y) \log\{f_{Y|X}(y \mid x)\} dxdy \quad \textbf{Equation 2}$$

We note that in the systems considered in this paper, X, representing the transmitted symbols, will always be a discrete random variable. On the other hand, the received signal represented by Y will be discrete or continuous depending on the considered case. For discrete random variables, the corresponding integral in both equations above is simply replaced by a summation over all values realized by that random variable.

Entropy is a mathematical representation of the a priori uncertainty of a random variable whereas Conditional Entropy represents the a posteriori uncertainty.

The *mutual information*, denoted I(X;Y), between X and Y is given by the following equation:

$$\mathrm{I(X\,;Y) = H(Y) - H(Y \mid X)}$$

We recall that entropy represents the uncertainty in a random variable. Therefore H(Y|X) represents the uncertainty in Y one X is known and we may conclude that the right hand term being the difference between H(Y) and H(Y|X) represents the amount of uncertainty that is *removed* from Y by the knowledge of X.

2.2 Shannon Channel Coding Theorem

Let X be a random variable representing the transmitted symbols. Let Y be a random variable representing the received signal, then the capacity of a channel is given by:

$$C = \max_{P(X)} I(X,Y) = \max_{P(X)} \{H(Y) - H(Y \mid X)\}$$

where the subscript P(X) indicates that the maximum is taken over all possible distributions or probability mass functions of the transmitted symbols represented by the random variable X.

Intuitively, the capacity of the channel is given by that input signal distribution which maximizes the mutual information between X and Y. In other words, the capacity is simply given by the maximum uncertainty that can be removed from Y by the knowledge of X.

2.3 Possible Parameterization of Capacity

Referring back to equations 1 and 2, we note that there are a few quantities that need to be defined in order to compute the capacity of a communications channel, namely the probability mass function $\mathbf{p_X(x)}$, the marginal probability density $\mathbf{f_Y(y)}$, the joint probability density $\mathbf{f_{X,Y}(x,y)}$ and finally the conditional density $\mathbf{f_{Y|X}(y|x)}$.

We note however that knowledge of $\mathbf{p_X(x)}$ and $\mathbf{f_{Y|X}(y|x)}$ is sufficient to characterize the remaining densities by an application of the law of total probability and Bayes' law shown below:

$$f_Y(y) = \sum_X p_X(x) f_{Y|X}(y \mid x) \qquad \textbf{Law of Total Probability}$$

$$f_{X,Y}(x,y) = f_{Y|X}(x \mid y) p_X(x) \qquad \textbf{Bayes' Law}$$

This parameterization greatly simplifies the complexity of calculating the capacity of a communications channel and will be used throughout this paper. Indeed, $\mathbf{f_{Y|X}(y|x)}$, generally referred to as Channel Transition Probabilities, is completely defined by the assumed characteristic of the channel. On the other hand $\mathbf{p_X(x)}$, the input symbol probabilities, is also easily defined, since it is discrete.

3 CAPACITY OF SYNCHRONOUS CDMA: PART I

As it was seen before, the CDMA multiple access techniques operates chip wise, adding corresponding chips in the spread sequences together, i.e. the received spread sequence's i^{th} chip is the sum of all the i^{th} chips of the users' spread sequences. Thus the operation on chips can be seen to operate independently and in parallel. This assumption is not completely accurate in the sense that there does exist inter-symbol interference, resulting from pulses leaking out of their respective time slots.

Nonetheless it will be assumed from this point forward that no such interference occurs. We therefore assume throughout this paper that our channel achieves chip synchronous decoding; the receiver is able to correctly identify the sum of corresponding chips of each user. Hence, under this assumption, the chip wise operation of CDMA characterizes the system completely. The per-chip capacity will be calculated, which amounts to finding out how many information bits can be correctly identified from a single chip or time-slot in a CDMA system.

In this section, the results will be presented for the calculation of the capacity per chip for a CDMA system with K users, with two underlying assumptions:

(1). Users are transmitting in BPSK modulation with equal powers.

(2). Unbiased probabilities in the transmitted symbols.

3.1 Noise free environment

In this sub-section it is further assumed that the channel is noise free. First, the chip-wise operation of CDMA in a system with two users is considered, for simplicity. This is depicted in figure 4 below:

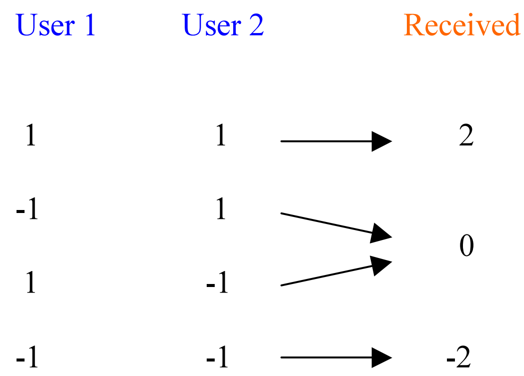


Figure 4. Additive channel model for CDMA in a noise free environment

We note that figure 4 depicts nothing more than a simple additive channel, whose capacity can be easily analyzed and generalized for a system with k users.

Let X be a random variable representing the transmitted symbols and therefore taking value on the set of vectors $\{(x_1, x_2, \ldots, x_k)\}$ of order 2^k , where $x_i = \pm 1$. Let Y be a random variable representing the received symbols, and hence taking value on the set $\{2i - k\}_{i=0}^{k}$, whose elements correspond to a distinct sum of coordinates of the vectors X.

The capacity of this channel is given by:

$$C_{chip} = \max_{P(X)} [H(Y) - H(Y|X)]$$

Recalling assumption (2) above and considering the additive channel model of figure 4, it is noted:

1. Each user is equally likely to transmit a +1 or a –1; therefore X is a discrete uniformly distributed random variable

$$P(X = x) = \frac{1}{2^k} \quad \forall \; x$$

   Therefore, there is no need to maximize over P(X) since the latter is uniform and fixed.

2. $P(Y \mid X = x) = \begin{cases} 1 \\ 0 \end{cases} \quad \forall \; x.$

In other words, knowledge of X makes knowledge of Y certain; this can be seen by the fact that there is only one arrow from a given value of X to the set of values of Y, and hence only one possibility of Y for each element of X. Hence Y|X is a deterministic random variable with H(Y|X)=0.

The expression for capacity for our additive channel or, equivalently, the capacity per chip reduces to:

$$\mathrm{C}_{\mathrm{chip}} = \mathrm{H}(\mathrm{Y}),$$

where knowledge of P(Y) is necessary and sufficient. Recalling equation (1) and our observations above:

$$P(Y) = \sum_{X} P(Y \mid X = x)P(X = x) = \frac{1}{2^k}\sum_{x} P(Y \mid X = x) \qquad \textbf{Equation 3}$$

Since $P(Y|X=x) = 1$ *or* 0, the summation simply represents a count of the arrows coming into $Y=y_i$. Equivalently, the summation represents the number of symbols x whose coordinates sum to the same value y_i, belonging to the set $\{2i-k\}_{i=0}^{k}$.

It can be easily seen that all the symbols $x=(x_1, x_2, \ldots, x_k)$ summing to the same $y_i = 2i-k$ are characterized as having $(k-i)$ coordinates of value +1 and, conversely, (i) coordinates of value –1. Therefore, the number of such x symbols, or the above summation term in equation (3) for $Y = y_i = 2i-k$, is simply given by

$$\binom{k}{i} = \binom{k}{k-i}$$

and thus

$$P(Y = y_i) = \frac{1}{2^k}\binom{k}{i}$$

Therefore, the following equation yields the per-chip capacity for a CDMA system with k users:

$$C_{chip}^{k} = H(Y)^{k} = -\frac{1}{2^{k}}\sum_{i=0}^{k}\binom{k}{i}\log_{2}\left[\binom{k}{i}\cdot\frac{1}{2^{k}}\right]$$

The results, obtained using Matlab are shown in figure 5 below:

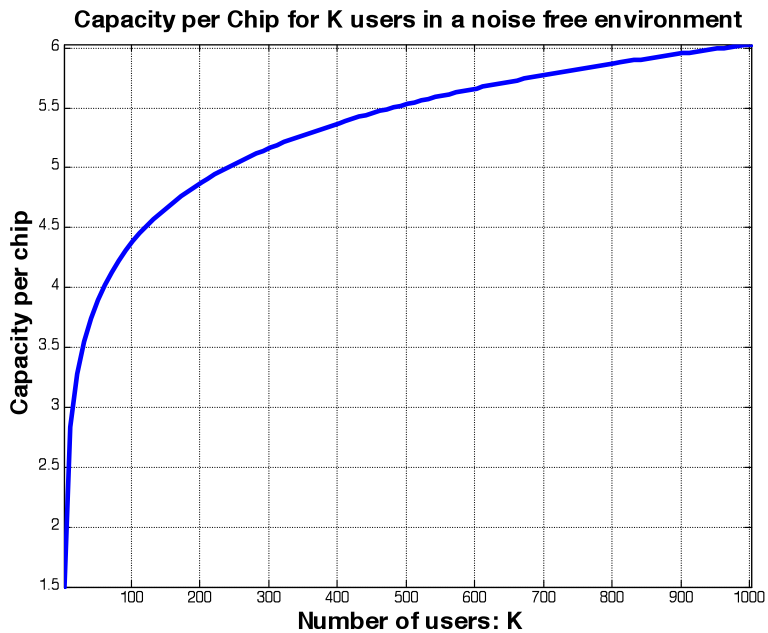


Figure 5. Capacity per chip without noise

It is noted that the capacity per chip increases logarithmically with the number of users. If an average CDMA cell is considered with approximately 200 associated users, we note that one is able to transmit up to 5 information bits per chip or time slot, thereby showing an increase of a factor of 5 with respect to the capacity of a TDMA system. This is an extremely encouraging result with respect to the capacity of a CDMA channel.

3.2 Adding white Gaussian noise

Here we maintain the assumption that BPSK modulation is used and that all users are transmitting with equal powers with unbiased symbol probabilities. We however forgo the unrealistic assumption that the environment is noise free. To this end we will consider that our additive channel model representing the chip wise operations for CDMA contains Additive White Gaussian Noise (AWGN).

In such a channel, X is still discrete and uniformly distributed, such that $\mathbf{p_X(x)}=1/2^k$. On the other hand, $\mathbf{f_{Y|X}(y|x)}$ is now completely determined by the distribution of the noise. The distribution is Gaussian with an assumed variance and a mean ,y, equal to the sum of the coordinates of x; i.e. belonging to the set $\{2i-k\}_{i=0}^{k}$.

The capacity of the channel is then given by

$$C = H(Y) - H(Y|X)$$

where all necessary parameters have been defined above.

3.2.1 CALCULATION OF H(Y|X)

Note that X is discrete while Y is now a continuous random variable. Therefore the equation for the Conditional Entropy H(Y|X) is now:

$$H(Y|X) = -\sum_X \int_{-\infty}^{\infty} f_{X,Y}(x,y) \log\{f_{Y|X}(y|x)\} dy$$

Using Bayes' Law on the joint density:

$$H(Y|X) = -\sum_X \int_{-\infty}^{\infty} p_X(x) f_{Y|X}(y|x) \log\{f_{Y|X}(y|x)\} dy$$

Substituting in $p_X(x)=1/2^k$ and recalling that $f_{Y|X}(y|x)$ is Gaussian-distributed with assumed variance $\square^2$ and mean $\square_x$, equal to the sum of the coordinates of x:

$$H(Y|X) = -\frac{1}{2^k}\sum_X \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu_x)^2}{2\sigma^2}} \log\left\{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu_x)^2}{2\sigma^2}}\right\} dy$$

We note that the integral is constant over different values of x since we are essentially integrating shifted values of the same function over (-∞, ∞). Let c represent the value of the integral:

$$H(Y \mid X) = -\frac{1}{2^k}\sum_X c = -c$$

since there are 2^k different x symbols. Now, setting the logarithm to a natural base and expanding it:

$$H(Y \mid X) = -\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu_x)^2}{2\sigma^2}} \left\{ \ln\left(2\pi\sigma^2\right)^{\frac{1}{2}} - \frac{(y-\mu_x)^2}{2\sigma^2} \right\} dy$$

The equation above essentially splits into two integrals: the first being a constant multiplied by the pdf of Y and therefore is equal to that constant; the second is seen to equal the variance □^{2} of Y|X multiplied by 1/2 □^{2}. Therefore,

$$H(Y \mid X) = \frac{1}{2}\ln\left(2\pi\sigma^2\right) + \frac{1}{2}$$

Converting to a base 2 logarithm [1]:

$$H(Y \mid X) = H(\sigma) = \frac{1}{2}\log_2\left(2e\pi\sigma^2\right)$$

3.2.2 CALCULATION OF H(Y)

The only remaining term to be defined in order to obtain the capacity of the communication channel under noisy conditions is H(Y). Recall that Y here is a continuous random variable whose entropy was earlier defined as:

$$H(Y) = -\int_{-\infty}^{\infty} f_Y(y)\log\{f_Y(y)\}dy$$

and $f_Y(y)$ is given by the Law of total probability:

$$f_Y(y) = \sum_X p_X(x) f_{Y|X}(y \mid x)$$

Using our predetermined parameters $\mathbf{p_X(x)}$ and $\mathbf{f_{Y|X}(y|x)}$, we get

$$f_Y(y) = \frac{1}{2^k}\sum_{i=0}^{k}\binom{k}{i}\frac{1}{\sqrt{2\pi\sigma^2}}.e^{\frac{-(y-2i+k)^2}{2\sigma^2}}$$

Thus H(Y) is completely defined. The next few graphs show $f_Y(y)$ for the communication channel with different numbers of users.

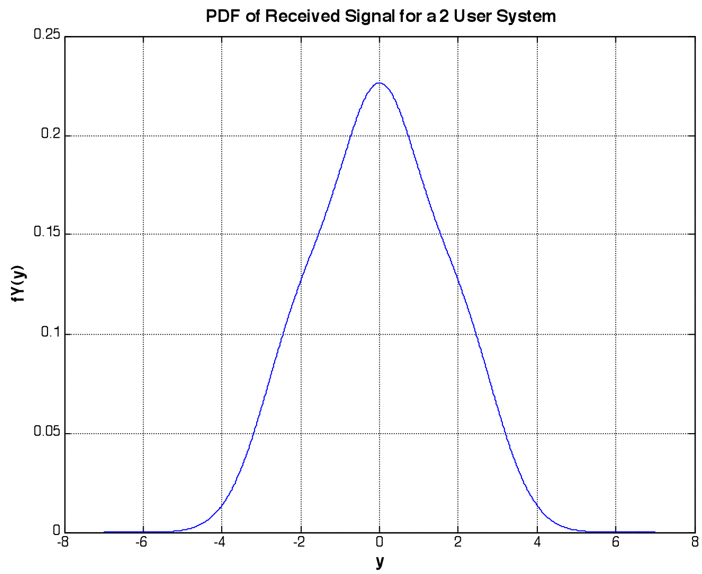


Figure 6. $f_Y(y)$ for a 2 User System

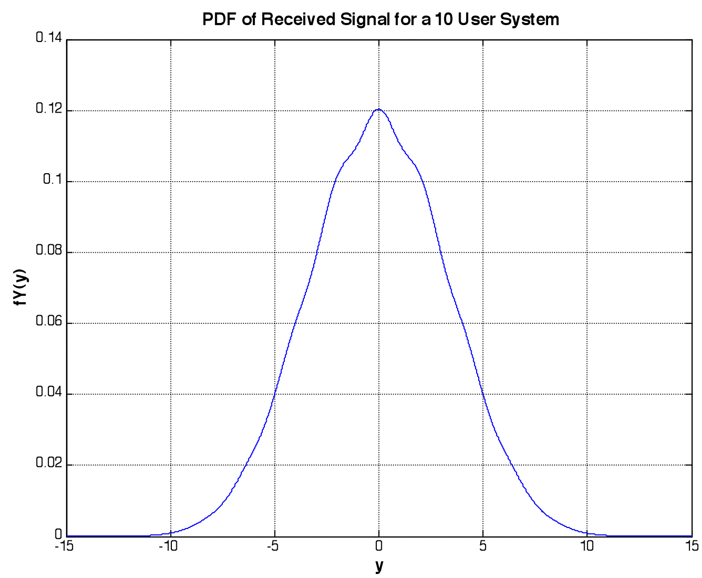


Figure 7. fY(y) for a 10 User System

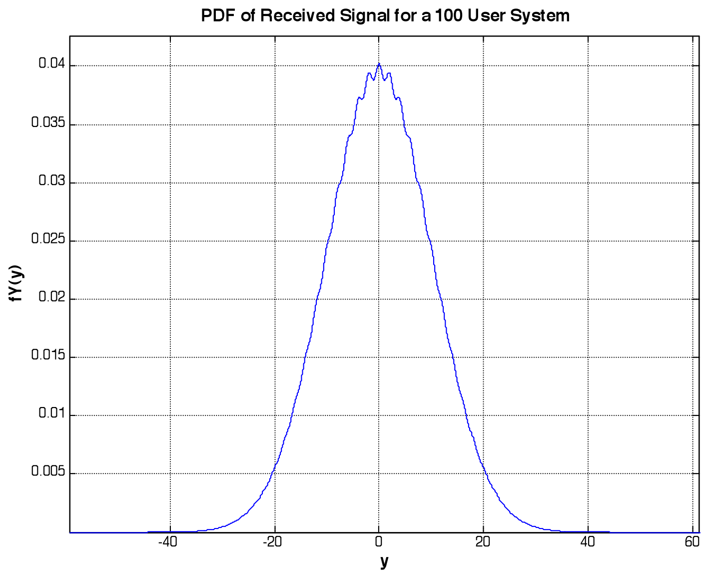


Figure 8. fY(y) for a 100 User System

3.2.3 CALCULATION OF CAPACITY

The capacity of our CDMA channel under Additive White Gaussian Noise was subsequently computed for different noise variances and is shown below.

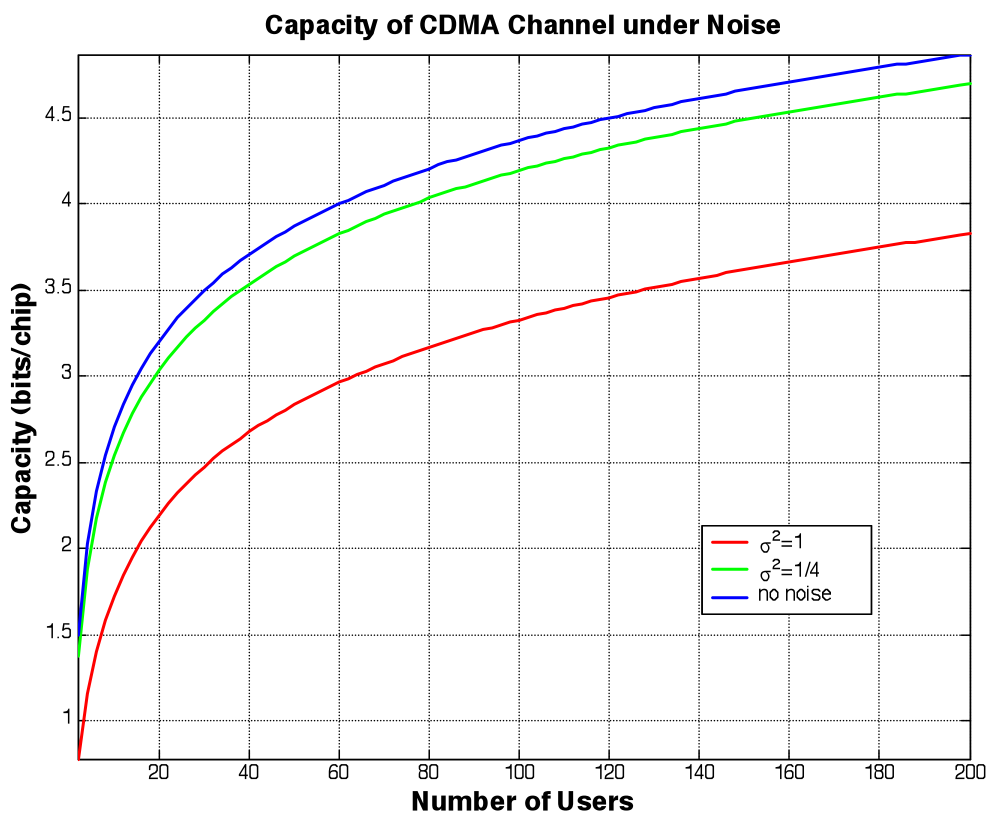


Figure 9. Capacity of CDMA Channel under Noise

We notice that, even with the addition of noise, the capacity per chip will still increase in a logarithmic fashion with the number of users. However it is lower than the noise free case, as expected.

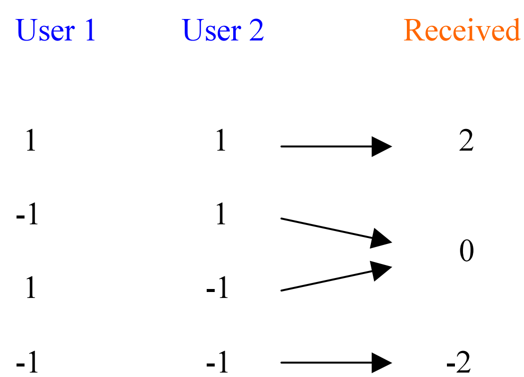
4 CAPACITY OF SYNCHRONOUS CDMA: PART II

In the earlier section, it was assumed that all users are transmitting in BPSK modulation with equal powers of 1 (i.e. amplitudes of ±1). In this section we will forgo our *equal user powers* assumptions while maintaining the assumption that users are transmitting their symbols (positive vs. negative amplitude) with equal probability.

4.1 Intuition

The underlying idea here is that capacity of the communication channel may increase if users purposely transmit their symbols with unequal powers or amplitudes. This can easily be seen below.

We recall our equal powers system (2 users) modeled by the simple additive channel:



The capacity of such this system in a noise free environment was calculated to 1.5 bit/chip. The loss of 0.5 bits is caused by the receiver's inability to distinguish each user's transmitted symbols when a 0 is received. If, on the other hand, User2 were to transmit using amplitudes of ±0.5, the resulting system, which is modeled below, clearly has a capacity of 2 bits/chip when there is no noise:

User 1	User 2		Received
1	0.5	→	1.5
1	-0.5	→	0.5
-1	0.5	→	-0.5
-1	-0.5	→	-1.5

Indeed, it can be clearly seen that there is never a resulting confusion at the receiver since the latter can always determine each user's sent symbols.

In this section, we will outline our results in considering the possibility of increasing channel capacity by assigning unequal powers to the users.

4.2 Noise free environment

In a noise free environment, considering the above discussion, the capacity calculations for such a system become trivial. Indeed, in a k user system, if the users amplitudes (or powers) are chosen properly – such that all transmitted symbol vectors $X=(x_1, x_2, \ldots, x_k)$ have distinct sums of coordinates – capacity is simply given by:

$$C_{chip}^{k} = k$$

Thus capacity increases linearly with the number of users.

4.3 Adding white Gaussian noise

In order for our analysis to have any realistic meaning we must include noise in the communications channel. The noise will be once again assumed to be white and Gaussian with a variance $\square^2$.

4.3.1 ASSIGNING DISTINCT POWERS TO USERS

To compare any two systems, equal amounts of power must be consumed by both. Therefore the following scheme was developed to assign users their respective powers:

1. User's transmitted amplitudes were separated from each other by a constant ratio specified as T dB. In other words, user's powers were spread by 2T dB.

2. The total user power was kept the same as in the equal power case: k units.

This assignment scheme can be achieved for any value of T or k as is seen below.

Let A represent the lowest amplitude transmitted. Let $t=10^{T/20}$. Conditions one and two translate to the following equation:

$$\sum_{i=0}^{k-1}\left(At^{i}\right)^{2} = A^{2}\sum_{i=0}^{k-1} t^{2i} = k$$

where t – specified by T – and k are known, and A is found to be

$$A = \left(\frac{k}{t^{2(k-2)} - 1 \Big/ t^{2} - 1} \right)^{\frac{1}{2}}$$

4.3.2 SPREADING USER AMPLITUDES BY 1- 2 DB

All of our previous development hold here, with the difference that X takes values on the set $\{(x_1, x_2, \ldots, x_k)\}$ of order 2^k and where $x_1=\pm A$, $x_2=\pm At$,…, $x_k=\pm At^{k-1}$.

Indeed, H(Y|X) is still constant with respect to the number of users and is a simple function of $\square^2$. H(Y) changes in the sense that y_i no longer belongs to the set {2i-k}, but still takes a value in the set of distinct sums of coordinates of X.

In the following graph, user's amplitudes were spread by 1 dB and the resulting capacity was plotted along with the equal power case, for several noise variances.

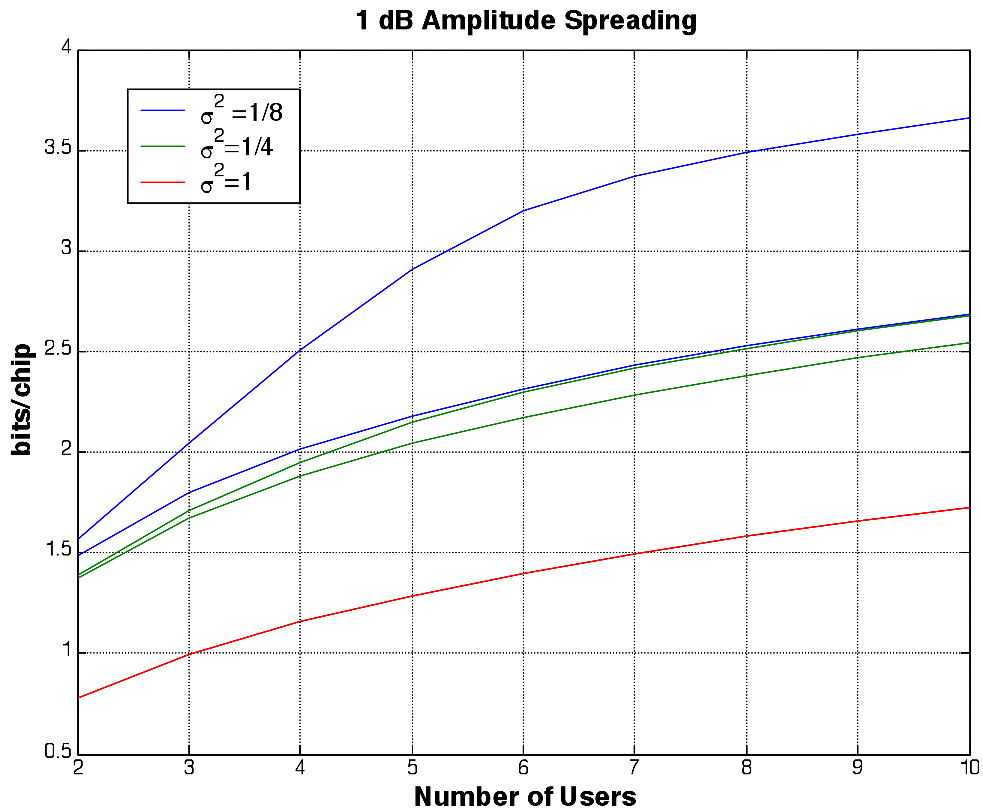


Figure 10. 1 dB Amplitude Spreading

The curves above should be read in pairs of the same color, where the higher curves represents the capacity after the user amplitudes were spread by 1 dB and the lower curve represent the capacity in the equal power case. It can be clearly seen that there is a significant gain in capacity for low noise variances: for $\square^2=1/8$. For example, a 10 users system experiences a significant gain of 1 bit/chip. However this gain seems to decrease exponentially with increasing $\square^2$: for $\square^2=1/4$, the gain has dropped to about 0.25 bits/chip whereas for $\square^2=1$, the gain is nonexistent.

The following graphs plots the capacity gain vs. SNR with the same 1 dB amplitude spread, as well as with a 2dB spread, for a 5 user system.

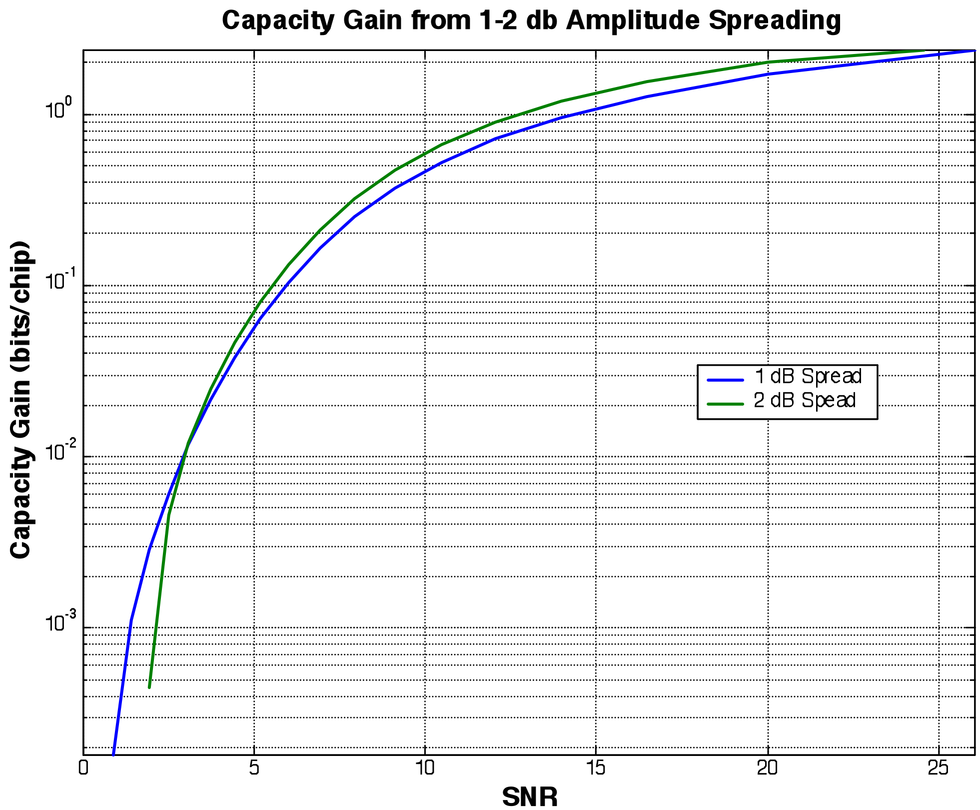


Figure 11. Capacity Gain from 1-2 dB Amplitude Spreading in a 5 User System

We can clearly see the exponential drop of gain in capacity vs. decreasing SNR. Indeed, the capacity gain starts off at around 1 bit for high SNR, falls quickly to 0.1 bits at mid SNR and finally to 0 bit at low SNR. We also note that a 2dB amplitude spread (4 dB power spread) adds little in terms of the capacity gain. The fact that the 2 dB spreading achieves a lower capacity gain than the 1 dB spreading, at 2.5 dB SNR, is attributed to round up error.

4.3.3 JUSTIFICATION OF EXPONENTIAL DROP OF CAPACITY GAIN

The justification concerning the exponential drop of capacity gain with decreasing SNR is best served with a simple example.

Consider a 3-user system where the user amplitudes have been spread by 1dB. The aforementioned amplitudes are given as: $A_1=\pm\ 0.8834$, $A_2 = \pm 0.9912$ and $A_3 = \pm 1.1122$.

Recalling our additive channel model with all the notation introduced, the graph below shows a plot of $f_{Y|X}(y|x)$ for every x belonging to the set $\{(\pm A_1, \pm A_2, \pm A_3)\}$ with a noise variance of 1.

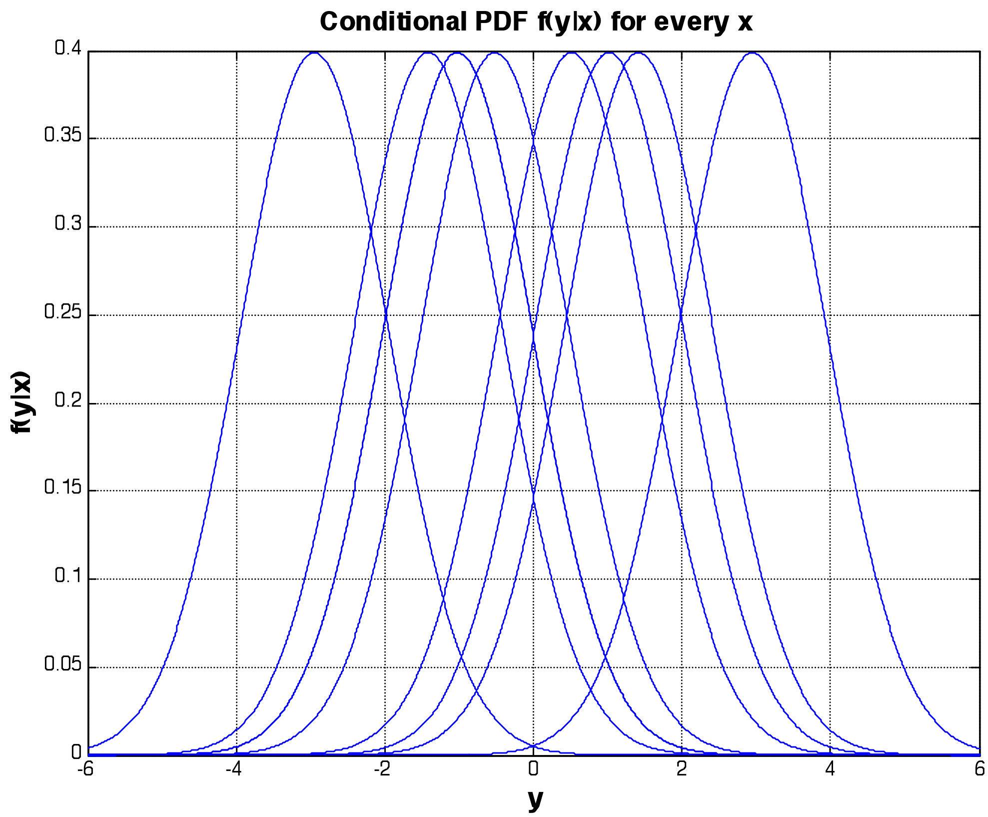


Figure 12. Conditional PDF's f(y|x) for evry x with σ^2=1

We note that although each x will have a distinct sum of coordinates, some of these sums turn out to be rather close to each other. Superimposing noise with a high variance around these sums renders them virtually indistinguishable from each other. Indeed, by observing the graph, we note that the channel under high noise closely resembles the equal power case where the only distinct sums of the coordinates of x , i.e. y_i , are found at $\{2i-k\}_{i=0}^{k}$ ={-3, -1, 1, 3}.

Thus, to conclude this section, assigning unequal user powers with the aim of increasing capacity is only viable at very high SNR and is therefore clearly not worth the cost since the capacity gain deteriorates exponentially with decreasing SNR.

5 CAPACITY OF SYNCHRONOUS CDMA: PART III

The aim of this section is the verification of our assumption that the capacity of our communication channel is maximized for unbiased input symbols, i.e. with a uniformly distributed random variable X.

Indeed this assumption was based on the easily proven theorem that for a symmetric channel, the input symbol pmf that maximizes capacity is a uniform pmf. However, our channel here is not symmetric and the assumption therefore must be verified. This will be done in a noisy environment, since the results would be easily carried over to the noise free case.

5.1 Biased Symbol Probabilities

We assume once again equal user powers. We let q be the probability of users transmitting +1 and, conversely, $1-q$ is the probability of transmitting –1.

We recall some notation introduced is section 4. X is a random variable representing the transmitted symbols and taking value on the set of vectors $\{(x_1, x_2, \ldots, x_k)\}$ of order 2^k, where $x_i=\pm1$.

Y is a random variable representing the received symbols, and hence, taking value on the set $\{2i-k\}_{i=0}^{k}$, whose elements correspond to distinct sums of coordinates of the vectors X.

All of the previous developments from section 4 remain valid, with the exception that X is now binomially distributed. Indeed for a given q, $p_X(x)=q^u(1-q)^{k-u}$, where u is the number of coordinates of x equal to +1.

In this context, we note that H(Y|X) is still defined by the same expression:

$$H(Y\mid X)=H(\sigma)=\frac{1}{2}\log_2\left(2e\pi\sigma^2\right)$$

H(Y) is now defined by a new $f_Y(y)$, taking account of the binomial distribution of X:

$$f_Y(y) = \sum_{i=0}^{k} \binom{k}{i} q^{k-i} (1-q)^i \frac{1}{\sqrt{2\pi\sigma^2}} . e^{\frac{-(y-2i+k)^2}{2\sigma^2}}$$

Recall here that $y_i = 2i\text{-}k$ corresponds to the received signal for all x with *i* coordinates equal to −1.

The following graph shows capacity as a function of *q* for channels with different numbers of users.

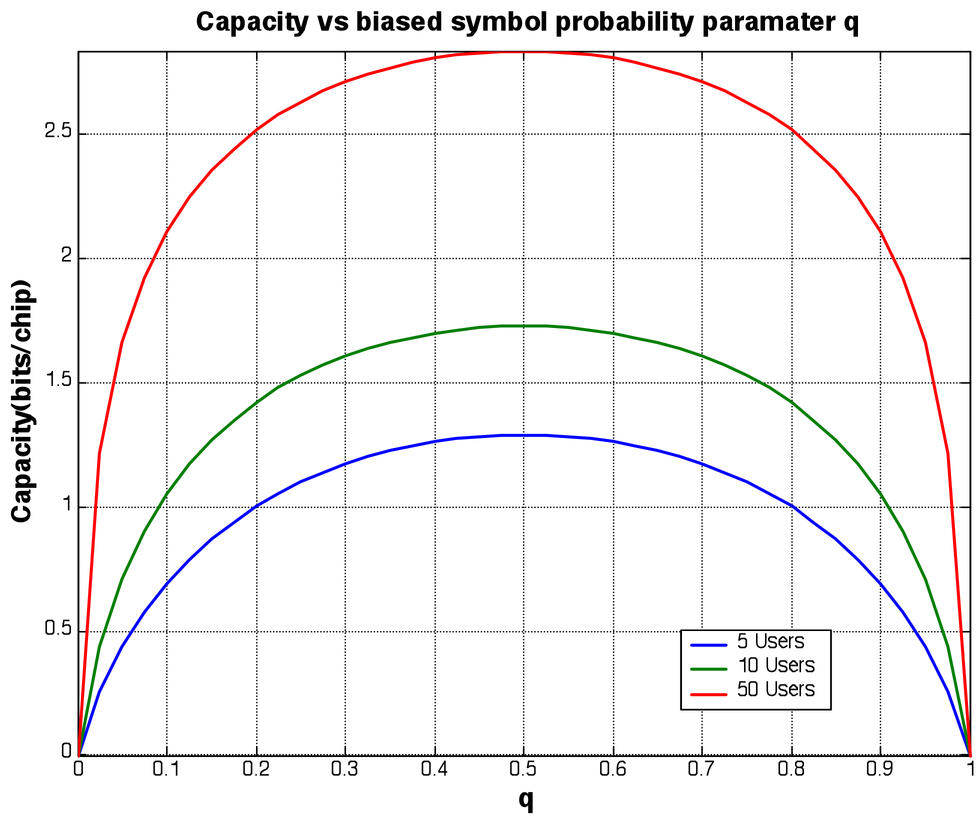
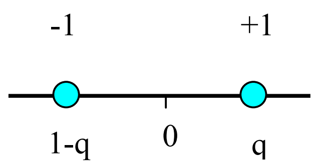


Figure 13. Capacity vs biased symbol probability parameter q at 0 dB SNR

The capacity is clearly seen to be maximized for *q*=0.5, corresponding to the previous unbiased symbol probability assumption. However as will be seen in the next section, these plots are not valid in the sense that for different *q* parameters, users' energies are not equal.

5.2 Centering users' constellations

We begin by examining each user's constellation when transmitting with biased symbols.



The average energy here can easily be calculated to be

$$\overline{E} = \sum E_i P_i = 1$$

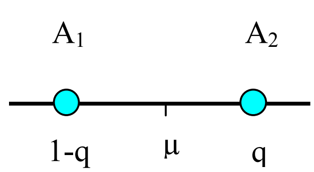
and the mean of the above constellation is calculated to be

$$\mu = 2q - 1$$

Therefore it is possible to center each user's constellation about the mean, which results in a reduction of average energy:

$$\overline{E_{new}} = 1 - \mu^2$$

This requires a subsequent scaling up of the amplitude in order to restore an average energy of 1.



Where A_1 and A_2 are given by

$$A_1 = \frac{2(1-q)}{\sqrt{1-\mu^2}} \qquad A_2 = \frac{-2q}{\sqrt{1-\mu^2}}$$

By applying the above transformation on the users' amplitudes, the average energy will always be equal to 1, regardless of the parameter *q*, yielding more valid results. The following graph plots the capacity for a 5-user system as a function of *q,* with constellation centering.

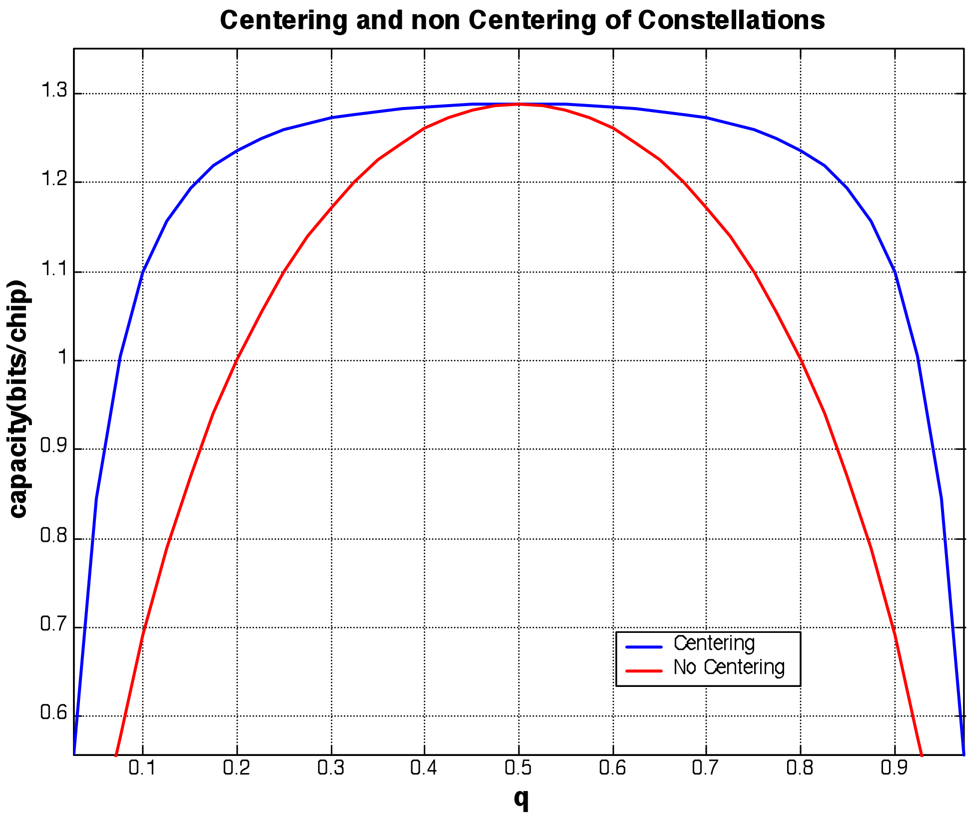


Figure 14 Centering User's Constellations for a 5 user system at 0 dB SNR

We note from the above graph, that even with constellation centering, capacity is still maximized at q=0.5. It is interesting to note that there is a significant gain in capacity for other *q* parameters. This was expected, since the centering of the constellations required us to scale up users' powers, thereby increasing the distance between the set of received values y_i . This allows for a greater distinction among the received signals.

6 SUMMARY OF RESULTS

Results from section 4.1 and 4.3 are summarized in the following graph showing the fundamental limit on capacity for synchronous CDMA at 0 dB SNR.

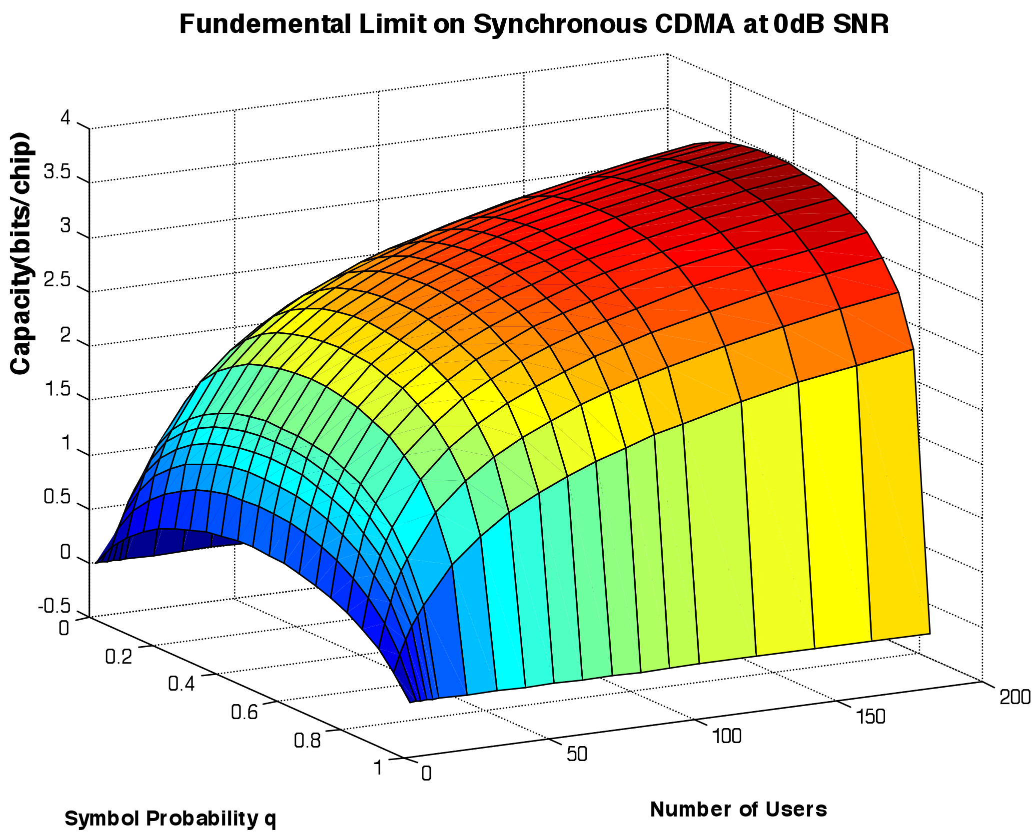


Figure 15. Fundemental Limit on Synchronous CDMA Transmission at 0dB SNR

We note the logarithmic increase of capacity versus number of users as well as the dome-like structure maximized at q=0.5.

CONCLUSION

The per-chip capacity of synchronous CDMA has been calculated, establishing the fundamental limit on the rate of transmission for synchronous CDMA communication channels. The three main results were: first, that with equal user powers, capacity increases logarithmically with the number of users in both noise free and noisy environments; second, and most interesting, that purposely assigning unequal user powers achieves little capacity gain at mid or low SNR; and thirdl, that the input pmf which maximizes capacity is a uniform pmf.

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