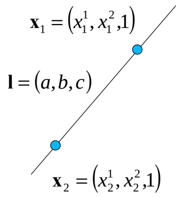
ESTIMATION

- 2D Lines and Ellipses
- 2D Projective transformations
- 3D to 2D Camera Projection
- Fundamental matrix computation

FITTING LINES



Points belong to line:

$$0 = l\mathbf{x}_1 = ax_1^1 + bx_1^2 + c$$

$$0 = l\mathbf{x}_2 = ax_2^1 + bx_2^2 + c$$

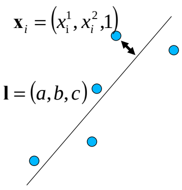
Additional constraint:

$$1 = \mathrm{c}$$

$$1 = \mathrm{a} + \mathrm{b} + \mathrm{c}$$

$$1 = \mathrm{a}^2 + b^2$$

FITTING LINES



Orthogonal distance to line:

$$h_i = \mathbf{l} \bullet \mathbf{x}_i = ax_i^1 + bx_i^2 + c \text{ if } a^2 + b^2 = 1$$

LINEAR ALGEBRA

Least squares minimization:

$$\text{Minimize} \sum_i (ax_i^1 + bx_i^2 + c)^2 \text{ wrt } a,b \text{ with } a^2 + b^2 = 1$$

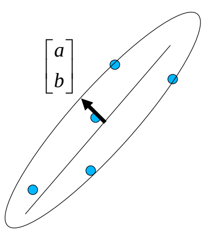
$$\sum_{\mathrm{i}} (ax_i^1 + bx_i^2 + c)^2 = \sum_i (ax_i^1 + bx_i^2)^2 + 2\,\mathrm{c}\,(\mathrm{a}\sum_{\mathrm{i}} x_i^1 + b\sum_{\mathrm{i}} x_i^2) + \sum_i c^2$$

Matrix formulation assuming centered coordinates:

$$\text{Minimize} \left\| \mathrm{M} \begin{bmatrix} \mathrm{a} \\ \mathrm{b} \end{bmatrix} \right\|^2 \text{ wrt } a,b \text{ subject to } a^2 + b^2 = 1 \text{ with } \mathbf{M} = \begin{bmatrix} x_1^1 & x_1^2 \\ \dots & \dots \\ \dots & \dots \\ x_n^1 & x_n^2 \end{bmatrix}$$

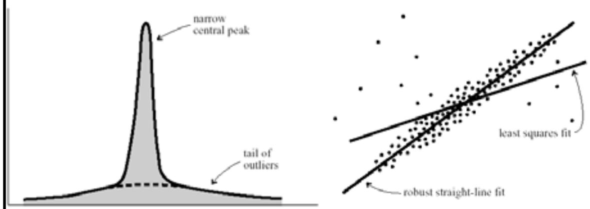
$$\Rightarrow \begin{bmatrix} \mathrm{a} \\ \mathrm{b} \end{bmatrix} \text{ is the eigenvector corresponding to the smallest value of } M^t M$$

MOMENTS



$$M^t M = \begin{bmatrix} \sum x_i^1 x_i^1 & \sum x_i^1 x_i^2 \\ \sum x_i^1 x_i^2 & \sum x_i^2 x_i^2 \end{bmatrix}$$

OUTLIERS



Outliers have a disproportionately large influence
→ They must be eliminated.

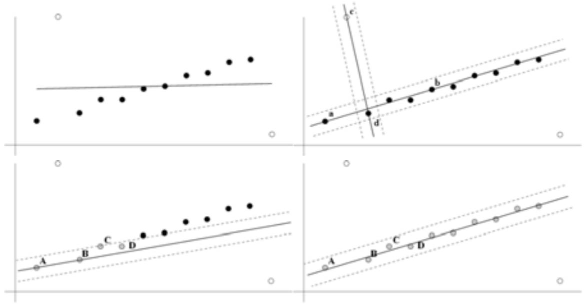
RANSAC

Given data set S:

1. Randomly select a minimal sample s of data points from S and instantiate the model M.
2. If too few points in S are “close enough” to M go back to 1.
3. Use the points that are “close enough” to robustly fit in the least-square sense and terminate.

Hartley, Chap 4.7

RANSAC



ROBUST LEAST-SQUARES

- Inliers are taken to be the points whose distance d_i to the model is less than t
- Minimize the robust criterion

$$\sum_i \gamma(d_i) \text{ with } \gamma(e) = \begin{cases} e^2 \text{ if } |e| < t \\ t^2 \text{ if } |e| \geq t \end{cases}$$

RANSAC PARAMETERS

- Distance threshold:

$$t^2 \propto \sigma^2$$

- Number of attempts:

$$N = \log(1-p)/\log(1-(1-\varepsilon)^s)$$

- Size of the consensus set:

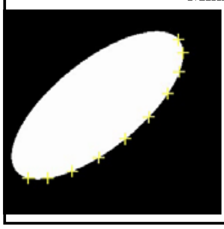
$$T = (1-\varepsilon)n$$

FITTING ELLIPSES

For each point:

$$d_a(\mathbf{x}_i, \mathbf{E}) = ax_i^{1^2} + bx_i^1 x_i^2 + cx_i^{2^2} + dx_i^1 + ex_i^2 + f$$
$$= \begin{bmatrix} x_i^{1^2} & x_i^1 x_i^2 & x_i^{2^2} & x_i^1 & x_i^2 & 1 \end{bmatrix} \bullet \mathbf{E}$$

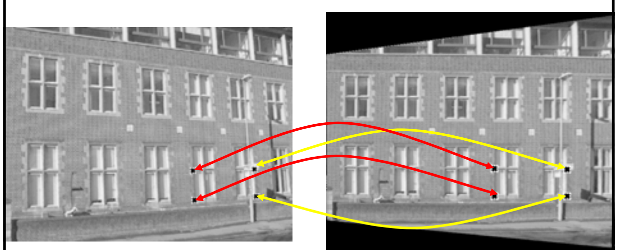
Minimize:



$$\sum_i d_a(\mathbf{x}_i, \mathbf{E})^2 = \|\mathbf{AE}\| \text{ subject to } \|\mathbf{E}\| = 1$$

$$\text{where } \mathbf{A} = \begin{bmatrix} x_1^{1^2} & x_1^1 x_1^2 & x_1^{2^2} & x_1^1 & x_1^2 & 1 \\ \Lambda & \Lambda & \Lambda & \Lambda & \Lambda & \Lambda \\ x_n^{1^2} & x_n^1 x_n^2 & x_n^{2^2} & x_n^1 & x_n^2 & 1 \end{bmatrix}$$

FROM CORRESPONDENCES TO HOMOGRAPHY



PARAMETERS OF A HOMOGRAPHY

$$\mathbf{x}' = \mathbf{H}\mathbf{x}$$

- Number of measurements required:
  - 8 degrees of freedom.
  - 2 constraints per correspondence.
- Linear solutions:
  - Minimal vs over-constrained solutions
  - Eliminating outliers
- Gold standard algorithm.

Hartley, Chap 4.

GOAL

Given $n \geq 4$ 2D to 2D point correspondences $\mathbf{x}_i \leftrightarrow \mathbf{x}_i'$, find the 2D Homography matrix $\mathbf{H}$ such that :

$$\forall i \ , \ \mathbf{x}_i' = \mathbf{H}\mathbf{x}_i \quad ,$$

or equivalently :

$$\forall i \ , \ \mathbf{x}_i' \times \mathbf{H}\mathbf{x}_i = 0 \quad .$$

DIRECT LINEAR TRANSFORM

Hartley, Chap 4.1

$$\text{Let } H = \begin{bmatrix} \mathbf{h}^{1T} \\ \mathbf{h}^{2T} \\ \mathbf{h}^{3T} \end{bmatrix} = \begin{bmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{bmatrix}$$

$$\mathbf{H}\mathbf{x}_i = \begin{bmatrix} \mathbf{h}^{1T}\mathbf{x}_i \\ \mathbf{h}^{2T}\mathbf{x}_i \\ \mathbf{h}^{3T}\mathbf{x}_i \end{bmatrix} \Rightarrow 0 = \mathbf{x}_i' \times \mathbf{H}\mathbf{x}_i = \begin{bmatrix} y_i'\mathrm{h}^{3T}\mathbf{x}_i - w_i'\mathrm{h}^{2T}\mathbf{x}_i \\ w_i'\mathrm{h}^{1T}\mathbf{x}_i - x_i'\mathrm{h}^{3T}\mathbf{x}_i \\ x_i'\mathrm{h}^{2T}\mathbf{x}_i - y_i'\mathrm{h}^{1T}\mathbf{x}_i \end{bmatrix}$$

Since $\mathbf{h}^{jT}\mathbf{x}_i = \mathbf{x}_i^T\mathbf{h}^j$, this can be written as :

$$\mathbf{A}_i\mathbf{h} = 0 \text{ with } \mathbf{A}_i = \begin{bmatrix} 0 & -w_i'\mathbf{x}_i^T & y_i'\mathbf{x}_i^T \\ w_i'\mathbf{x}_i^T & 0 & -x_i'\mathbf{x}_i^T \\ -y_i'\mathbf{x}_i^T & x_i'\mathbf{x}_i^T & 0 \end{bmatrix} \text{ and } \mathbf{h} = \begin{bmatrix} \mathbf{h}^1 \\ \mathbf{h}^2 \\ \mathbf{h}^3 \end{bmatrix}$$

DIRECT LINEAR TRANSFORM

$\mathbf{h}$ is solution of the set of linear equations :

$$\forall i, \mathbf{A}_i\mathbf{h} = 0 \text{ with } \mathbf{A}_i = \begin{bmatrix} 0 & -w_i'\mathbf{x}_i^T & y_i'\mathbf{x}_i^T \\ w_i'\mathbf{x}_i^T & 0 & -x_i'\mathbf{x}_i^T \end{bmatrix} .$$

The third row of $\mathbf{A}_i$ has been dropped because it is a linear combination of the first two.

Can be written as :

$$\mathbf{A}\mathbf{h} = 0, \text{ where } \mathbf{A} = \begin{bmatrix} \mathbf{A}_1 \\ \Lambda \\ \mathbf{A}_n \end{bmatrix} \text{ is a } (2n) \times 9 \text{ matrix.}$$

MINIMAL SOLUTION

With 4 correspondences we can write :

$$\mathbf{A}\mathbf{h} = 0, \text{ where } \mathbf{A} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \mathbf{A}_3 \\ \mathbf{A}_4 \end{bmatrix} \text{ is an } 8 \times 9 \text{ matrix.}$$

$\Rightarrow$ Unique solution with one additional constraint :

$h_9 = 1$

$\|\mathbf{h}\| = 1$

LEAST-SQUARES SOLUTION

With more than 4 correspondences we minimize :

$$\|\mathbf{A}\mathbf{h}\| \text{ subject to } \|\mathbf{h}\| = 1, \text{ where } \mathbf{A} = \begin{bmatrix} \mathbf{A}_1 \\ \Lambda \\ \mathbf{A}_n \end{bmatrix} \text{ is a } (2n) \times 9 \text{ matrix.}$$

$\Rightarrow$ $\mathbf{h}$ is the eigenvector corresponding to the smallest eigenvalue of $\mathbf{A}$.

GOAL

Given $n \geq 4$ 2D to 2D point correspondences $\mathbf{x}_i \leftrightarrow \mathbf{x}_i'$,
find the 2D Homography matrix $\mathbf{H}$ such that :

$$\forall i \ , \ \mathbf{x}_i' = \mathbf{H}\mathbf{x}_i$$

BASIC DLT ALGORITHM

- For each correspondence $\mathbf{x}_i \leftrightarrow \mathbf{x}_i'$, compute the matrix 2x9 matrix $\mathbf{A}_i$
- Assemble the $2n \times 9$ $\mathbf{A}$ matrix.
- Use singular value decomposition to compute $\mathbf{h}$, the eigenvector of $\mathbf{A}$ associated to the smallest eigenvalue.
- Rearrange the elements of $\mathbf{h}$ to form $\mathbf{H}$.

DEGENERACIES

Given 4 2D to 2D point correspondences $\mathbf{x}_i \leftrightarrow \mathbf{x}_i'$, is there always a single 2D Homography matrix $\mathbf{H}$ such that :

$$\forall i \ , \ \mathbf{x}_i' = \mathbf{H}\mathbf{x}_i \ ?$$

Answer : Not if $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are colinear.

DEGENERACIES

Assuming that $\exists \mathbf{l}, \ \ \mathbf{l} \bullet \mathbf{x}_i = \mathbf{l}^T \mathbf{x}_i = 0 \ \ \text{for } 1 \leq i \leq 3$,
let $\mathbf{H}^* = \mathbf{x}_4' \mathbf{l}^T$.

$$\mathbf{H}^* \mathbf{x}_4 = \mathbf{x}_4' (\mathbf{l}^T \mathbf{x}_4) = \mathbf{x}_4'$$
$$\mathbf{H}^* \mathbf{x}_i = \mathbf{x}_4' (\mathbf{l}^T \mathbf{x}_i) = \mathbf{0}$$

$\mathbf{x}_1', \mathbf{x}_2'$ and $\mathbf{x}_3'$ are colinear :
There is another solution $\mathbf{H}$ and any $\alpha\mathbf{H} + \beta\mathbf{H}^*$ is also a solution.
$\mathbf{x}_1', \mathbf{x}_2'$ and $\mathbf{x}_3'$ are not colinear :
$\mathbf{H}^*$ is the only solution and it is not a homography.

NON INVARIANCE

- Basic DLT is *not* invariant to similarity transformations of the image.
- Some coordinate systems are better than others to perform the computation.

→ Normalize, that is precondition, the data before running DLT.

NORMALIZATION

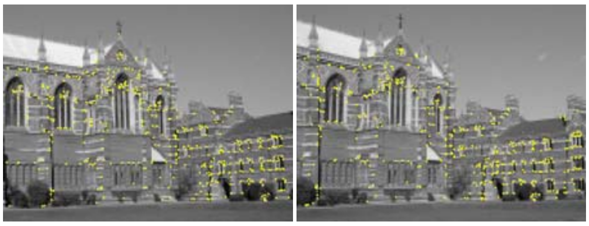
No normalization With normalization

In each image independently:
Translate the points to bring their centroid is at the origin.
Scale the points so that the average distance to the origin is sqrt(2).

NORMALIZED DLT ALGORITHM

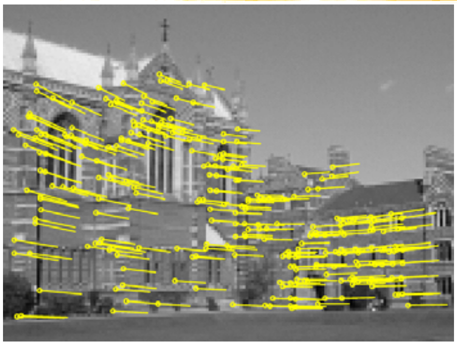
- Find a similarity transformation $\mathbf{T}$ that takes the $\mathbf{x}_i$s to the $\tilde{\mathbf{x}}_i$s such that their centroid is the origin and their average distance to it is $\sqrt{2}$.
- Find an equivalent similarity transformation $\mathbf{T}'$ that takes the $\mathbf{x}_i'$ s to the $\tilde{\mathbf{x}}_i'$ s.
- Apply basic DLT to the $\tilde{\mathbf{x}}_i \leftrightarrow \tilde{\mathbf{x}}_i'$ correspondences to obtain homography $\tilde{\mathbf{H}}$.
- Set $\mathbf{H} = \mathbf{T}'^{-1} \tilde{\mathbf{H}} \mathbf{T}$

DETECTED CORNERS



The camera rotates about its optical center

CORRESPONDENCES



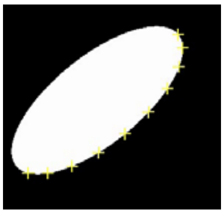
RANSAC

Given a set of correspondences S:

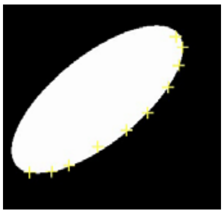
1. Randomly select a sample s of four correspondences from S and use DLT to instantiate H.
2. If too few correspondences in S are "good enough" to M go back to 1.
3. Use the correspondences that are "good enough" to robustly fit in the least-square sense and terminate.

Hartley, Chap 4.7

ELLIPSE FITTING REVISITED

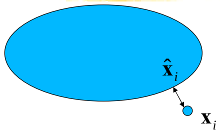


Perfect data



Noisy data

ALGEBRAIC vs EUCLIDEAN DISTANCES



For each point:

$$d_a(\mathbf{x}_i, \mathbf{E}) = a{x_i^1}^2 + b x_i^1 x_i^2 + c{x_i^2}^2 + d x_i^1 + e x_i^2 + f$$

$$d_e(\mathbf{x}_i, \mathbf{E}) = d(\mathbf{x}_i, \hat{\mathbf{x}}_i)$$

Minimize:

$$\sum_i d_e(\mathbf{x}_i, \mathbf{E})^2 \text{ instead of } \sum_i d_e(\mathbf{x}_i, \mathbf{E})^2$$

NON-LINEAR MINIMIZATION

$$\text{Minimize } \|F(\mathbf{E})\|^2, \text{ where } F(\mathbf{E}) = \begin{bmatrix} d_e(\mathbf{x}_1,\mathbf{E}) \\ \Lambda \\ d_e(\mathbf{x}_n,\mathbf{E}) \end{bmatrix}$$

Given the current value of $\mathbf{E}$, find $\mathbf{dE}$ that minimizes:

$$\frac{1}{2}\|F(\mathbf{E}+\mathbf{dE})\|^2 = \frac{1}{2}F(\mathbf{E}+\mathbf{dE})^T F(\mathbf{E}+\mathbf{dE})$$

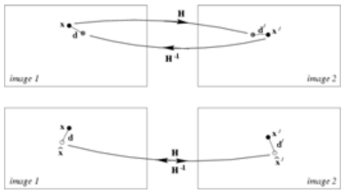
$$\Rightarrow 0 = \frac{\partial}{\partial \mathbf{E}}(F(\mathbf{E}+\mathbf{dE})^T F(\mathbf{E}+\mathbf{dE}))$$

$$= \mathbf{J}^T(F(\mathbf{E})+\mathbf{JdE}) \text{ with } \mathbf{J} = \begin{bmatrix} \frac{\partial F_1}{\partial E_1} & \Lambda & \frac{\partial F_1}{\partial E_m} \\ \Lambda & \Lambda & \Lambda \\ \frac{\partial F_n}{\partial E_1} & \Lambda & \frac{\partial F_n}{\partial E_m} \end{bmatrix}$$

At each iteration, solve:

$$\mathbf{J}^T\mathbf{JdE} = -\mathbf{J}^T F(\mathbf{E})$$

GEOMETRIC DISTANCES



Minimize:

$\sum_i d_e(\mathbf{x}_i^{'},\mathbf{H}\mathbf{x}_i)^2$ (Error in one image)

$\sum_i d_e(\mathbf{x}_i^{'},\mathbf{H}\mathbf{x}_i)^2 + \sum_i d_e(\mathbf{x}_i,\mathbf{H}^{-1}\mathbf{x}_i^{'})^2$ (Symmetric transfer error)

$\sum_i d_e(\mathbf{x}_i,\hat{\mathbf{x}}_i)^2 + \sum_i d_e(\mathbf{x}_i^{'},\hat{\mathbf{x}}_i^{'})^2$ subject to $\hat{\mathbf{x}}_i^{'} = \hat{\mathbf{x}}_i$

MODIFIED GOAL

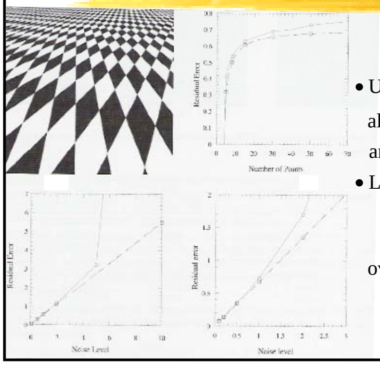
Given $n \geq 4$ 2D to 2D point correspondences $\mathbf{x}_i \leftrightarrow \mathbf{x}_i^{'}$,

find the 2D Homography matrix $\hat{\mathbf{H}}$ such that and the set of subsidiary points $\hat{\mathbf{x}}_i$ such that:

$$\sum_i \mathrm{d}(\mathbf{x}_i,\hat{\mathbf{x}}_i)^2 + \mathrm{d}(\mathbf{x}_i^{'},\hat{\mathbf{x}}_i^{'})^2$$

is minimized, where $\hat{\mathbf{x}}_i^{'} = \hat{\mathbf{H}}\mathbf{x}_i$.

GOLD STANDARD ALGORITHM



- Use the normalized DLT algorithm to initialize $\hat{\mathbf{H}}$ and take $\hat{\mathbf{x}}_i = \mathbf{x}_i$.
- LVM to minimize

$$\sum_i d(\mathbf{x}_i,\hat{\mathbf{x}}_i)^2 + d(\mathbf{x}_i^{'},\hat{\mathbf{x}}_i^{'})^2$$

over $\hat{\mathbf{H}}$ and $\hat{\mathbf{x}}_i$, i = 1K n.

ALGEBRAIC vs GEOMETRIC DISTANCES

Let us write:

$$\mathbf{x}_i^{'} = (x_i^{'}, y_i^{'}, w_i^{'})$$

$$\hat{\mathbf{x}}_i^{'} = (\hat{x}_i^{'}, \hat{y}_i^{'}, \hat{w}_i^{'}) = \mathbf{H}\mathbf{x}_i$$

$$\mathbf{A}_i\mathbf{h} = \begin{pmatrix} y_i^{'}\hat{w}_i^{'} - w_i^{'}\hat{y}_i^{'} \\ w_i^{'}\hat{x}_i^{'} - x_i^{'}\hat{w}_i^{'} \end{pmatrix}$$

Therefore :

$$d_a(\mathbf{x}_i^{'},\hat{\mathbf{x}}_i^{'}) = \sqrt{(y_i^{'}\hat{w}_i^{'} - w_i^{'}\hat{y}_i^{'})^2 + (w_i^{'}\hat{x}_i^{'} - x_i^{'}\hat{w}_i^{'})^2}$$

$$d_e(\mathbf{x}_i^{'},\hat{\mathbf{x}}_i^{'}) = \sqrt{\left(\frac{x_i^{'}}{w_i^{'}} - \frac{\hat{x}_i^{'}}{\hat{w}_i^{'}}\right)^2 + \left(\frac{y_i^{'}}{w_i^{'}} - \frac{\hat{y}_i^{'}}{\hat{w}_i^{'}}\right)^2}$$

$$= d_a(\mathbf{x}_i^{'},\hat{\mathbf{x}}_i^{'})/(w_i^{'}\hat{w}_i^{'})$$

CAMERA CALIBRATION

PARAMETERS OF PROJECTION MATRIX

$$\mathbf{x} = \mathbf{PX}$$

- Number of measurements required:
  - 11 degrees of freedom.
  - 2 constraints per correspondence.
- Direct linear transform:
  - Minimal solution for 6 correspondences
  - Over-constrained solutions by imposing

$$\|\mathbf{P}\| = 1 \text{ or } \mathbf{P}_{34} = 1$$

- Non linear optimization.

Hartley, Chap 7.

GOAL

Given $n \geq 6$ 3D to 2D point correspondences $\mathbf{X}_i \leftrightarrow \mathbf{x}_i$,
find the 3D projection matrix $\mathbf{P}$ such that :

$$\forall i \ , \mathbf{x}_i = \mathbf{PX}_i \quad ,$$

or equivalently :

$$\forall i \ , \ \mathbf{x}_i \times \mathbf{PX}_i = 0 \quad .$$

DIRECT LINEAR TRANSFORM

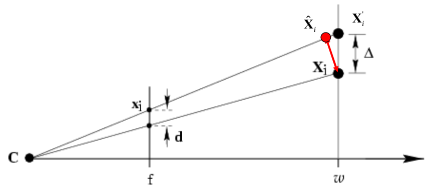
Let $\mathbf{P} = \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{bmatrix}$ where the $\mathbf{P}_i$ are 4 - vectors. We have

$$\forall i, \mathbf{A}_i \begin{pmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{pmatrix} = 0 \text{ with } \mathbf{A}_i = \begin{bmatrix} 0 & -w_i\mathbf{X}_i^T & y_i\mathbf{X}_i^T \\ w_i\mathbf{X}_i^T & 0 & -x_i\mathbf{X}_i^T \end{bmatrix} ,$$

which be written as :

$$\mathbf{A} \begin{pmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{pmatrix} = 0, \text{ where } \mathbf{A} = \begin{bmatrix} \mathbf{A}_1 \\ \Lambda \\ \mathbf{A}_n \end{bmatrix} \text{ is a } (2n)\times 12 \text{ matrix.}$$

ALGEBRAIC vs GEOMETRIC DISTANCES



The DLT algorithm minimizes the sum of squares of geometric distance between $\mathbf{X}_i$ and $\mathbf{X}'_i$ instead of $\hat{\mathbf{X}}_i$

NON-LINEAR MINIMIZATION

For $1 \leq i \leq n$

$$P_u(\mathbf{X}_i) = u_i + \varepsilon_{u,i}$$
$$P_v(\mathbf{X}_i) = v_i + \varepsilon_{v,i}$$

Minimize

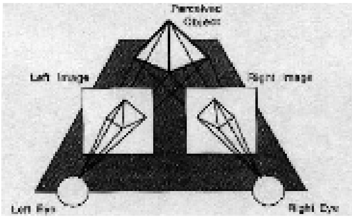
$$\sum_{i=1}^{n} \varepsilon_{u,i}^2 + \varepsilon_{v,i}^2 = \sum_{i=1}^{n} (P_u(\mathbf{X}_i) - u_i)^2 + (P_v(\mathbf{X}_i) - v_i)^2$$

with respect to the unknown calibration parameters.

GOLD STANDARD ALGORITHM

- Find a similarity transformation $\mathbf{T}$ that takes the $\mathbf{x}_i$s to the $\tilde{\mathbf{x}}_i$s such that their centroid is the origin and their average distance to it is $\sqrt{2}$.
- Find a similarity transformation $\mathbf{U}$ that takes the $\mathbf{X}_i$s to the $\tilde{\mathbf{X}}_i$s such that their centroid is the origin and their average distance to it is $\sqrt{3}$.
- Apply basic DLT to the $\tilde{\mathbf{X}}_i \leftrightarrow \tilde{\mathbf{x}}_i$ correspondences to obtain projection $\tilde{\mathbf{P}}$.
- Use $\tilde{\mathbf{P}}$ as a starting point to minimize geometric distance $\sum_i d_e(\tilde{\mathbf{x}}_i, \tilde{\mathbf{P}}\mathbf{X}_i)^2$.
- Set $\mathbf{P} = \mathbf{T}^{-1}\tilde{\mathbf{P}}\mathbf{U}$

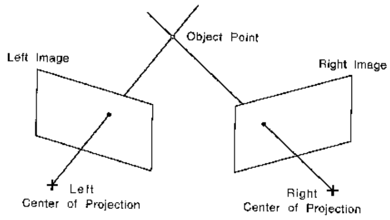
GEOMETRIC STEREO



Depth from two or more images:

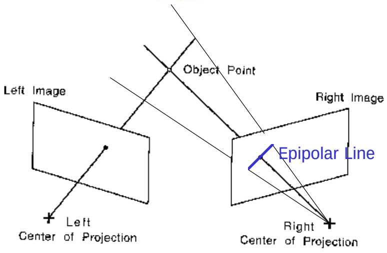
- Geometry of image pairs
- Establishing correspondences

TRIANGULATION



Geometric Stereo: Depth from two images

EPIPOLAR LINE



Line on which the corresponding point must lie.

EPIPOLAR LINES

Three points shown as red crosses.

Corresponding epipolar lines.

SHAPE FROM VIDEO

1. Treat consecutive images as stereo pairs.
2. Compute disparity maps.
3. Merge 3-D point clouds.
4. Represent as particles.

FUNDAMENTAL MATRIX

There is 3×3 matrix F such that for all corresponding points $\mathbf{x} \leftrightarrow \mathbf{x}'$

$$\mathbf{x}' = \mathbf{F}\mathbf{x}.$$

Therefore, the epipolar line corresponding to $\mathbf{x}$ is $\mathbf{l} = \mathbf{F}\mathbf{x}$.

Given a set of n point matches, we write

$$\begin{bmatrix} u_1'u_1 & u_1'v_1 & u_1' & v_1'u_1 & v_1'v_1 & v_1' & u_1 & v_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_n'u_n & u_n'v_n & u_n' & v_n'u_n & v_n'v_n & v_n' & u_n & v_n & 1 \end{bmatrix}$$

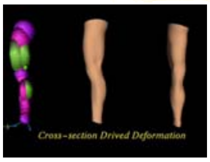
$\rightarrow$ DLT or non – linear minimization.

Hartley, Chap 9.

SHAPE & MOTION CAPTURE

COMPLEX 3-D MOTION

IMPLICIT SURFACES IN COMPUTER GRAPHICS



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M.P. Gascuel and A. Verroust and C. Puech. A Modeling System for Complex Deformable Bodies Suited to Animation and Collision Processing. *Journal of Visualization and Computer*, 1991.

D. Thalmann, J. Shen, and E. Chauvineau. Fast Realistic Human Body Deformations for Animation and VR Applications. In *Computer Graphics International*, June 1996.

The volumetric primitives melt like mercury drops

METABALLS: ELLIPSOIDAL PRIMITIVES

- Each one defines a field.

$$d_i(\mathbf{x}) = \mathbf{x}^T \cdot \mathbf{Q}_i^{\,T} \cdot \mathbf{Q}_i \cdot \mathbf{x}$$

$$f_i(\mathbf{x}) = e^{-2d_i(\mathbf{x})}$$

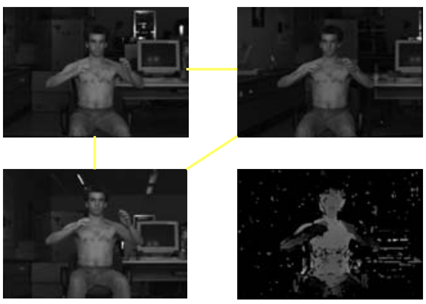
- The surface is an isosurface of their sums.

$$S = \{\mathbf{x} | F - T = 0\}, F(\mathbf{x}) = \sum_i^n f_i(\mathbf{x})$$

→ Algebraic distances of 3—D points to the surface can be computed without search and are differentiable.

→ Surface normals and curvatures can be computed both simply and exactly.

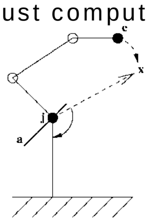
STEREO DATA



ROTATIONAL DERIVATIVES

To minimize: $F(\mathbf{x}, \Theta) - T \;\rightarrow\; \min$

Must compute:



$$\frac{\partial}{\partial\theta} F(\mathbf{x}, \Theta) = \frac{\partial}{\partial\theta} \sum_i^n f_i\big(d_i(\mathbf{x} \;, \Theta)\big)$$

$$\frac{\partial}{\partial\theta} d(\mathbf{x}, \Theta) = 2\; \mathbf{x}^T \cdot \mathbf{S}_\theta^T \mathbf{Q}_\theta^T \cdot \left[\frac{\partial}{\partial\theta} \mathbf{Q}_\theta \mathbf{S}_\theta\right] \cdot \mathbf{x}$$

$$\frac{\partial}{\partial\theta} \mathbf{Q}_\theta \mathbf{S}_\theta = \mathbf{Q}_\theta \mathbf{Rt}_{je} \overset{\rho}{\mathbf{a}}_{\mathbf{j}} \times (\mathbf{x} - \mathbf{j})$$